a, b = intermediate points along stream
c = cold end of counterflow exchanger
cc = cocurrent tube pass
cf = counterflow tube pass
h = hot end of counterflow exchanger
max = maximum obtainable
opt = economic optimum
1 = inlet of a stream
2 = outlet of a stream

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Dispersion in Layered Porous Media

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Experimental data on directional dispersion coefficients in transversely isotropic media, a special case of anisotropic media, were determined in models consisting of alternating layers of sintered glass and unconsolidated glass beads. Values of appropriate parameters to relate the velocity to the directional dispersion coefficient were determined for a variety of permeabilities and angles. Sensitivity of these parameters with respect to angle was determined, and results are presented which will permit estimate of when this effect is significant.

SCOPE

Porous media occur in many areas of chemical engineering, petroleum engineering, and hydrology. The usual difficulties with characterization of porous media occur because of their nonuniformity, heterogeneity, and anisotropy. Nonuniformity and heterogeneity are concerned with spatial variations of the media; anisotropy is concerned with variation with angle. To treat a general anisotropic medium is difficult; therefore, some special case of anisotropy is usually treated.

In this paper, we treat dispersion in transversely isotropic porous media, which are a special subset of anisotropic porous media. Transversely isotropic media characterize layered systems. Most petroleum and water reservoirs are layered. Many packed beds are packed in layers. The objective was to characterize the dispersion (mixing) of a fluid as it flows through such a medium as it relates to the angle the velocity makes with axis of isotropy and the velocity of flow. To take a single example, such knowledge is extremely important in the production of petroleum from underground deposits when this production is being accomplished by miscible displacement, that is, displacement of petroleum from the rock formation using a miscible fluid. The significant theory leading up to the treatment in this article is reviewed, and here we give results which permit the directional dispersion to be related both to velocity and to angle.

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CONCLUSIONS AND SIGNIFICANCE

The directional dispersion in models which approximate transversely isotropic porous media were fit by equations from a general theory for anisotropic porous media (Moranville, Kessler, and Greenkorn, 1977). The experiments show that for anisotropic media of practical significance, the number of measurements necessary to characterize the dispersion tensor is small enough to be tractable.

The experimental data presented represent a majority of the data available for a specific anisotropic system. There is need for additional data when flow is perpendicular to the axis of symmetry.

The results show that for real-world anisotropic systems, the variation of dispersion (mixing) with direction is significant. The system studied had a ratio of major to minor axis permeability of 8.6, which is representative of many natural and packed layered systems. The drift resulting from this anisotropic effect on mixing is significant and will cause errors in estimating miscible front movement and breakdown if ignored. Systems where the microscopic nature of the medium is important may give substantial directional effects, such as in tertiary oil recovery or movement of trace elements in porous formations.

The movement of various materials in miscible flow through porous media is of interest in many areas of chemical engineering (in adsorption and ion exchange columns, reactor engineering, and chromatography), petroleum engineering (in miscible displacement of oil with solvents and surface-active agents), hydrology (in movement of trace pollutants and salt water encroachment), soil physics (in movement of nutrients and pollutants), biophysics (in many life processes such as flow in the kidney). In order to predict movement of materials and locate fronts between miscible materials during flow in porous media, we need to model the process.

Almost all porous media possess, to some degree, nonidealities such as heterogeneity, nonuniformity, and anisotropy. It is our objective here to present experimental results regarding the effect of anisotropy (angular dependence) on dispersion. Specifically, we consider transversely isotropic porous media. These represent realistic porous media because most artificial beds of material are packed in layers, and, further, most naturally occurring media are layered. These layers cause directional effects on the pseudo transport coefficients, permeability and dispersion.

Dispersion as considered here is the macroscopic mixing of miscible fluids during laminar flow through a porous medium. When a traced fluid is forced through a porous medium initially filled with untraced fluid, the concentration change in the effluent (the breakthrough curve of the tracer) shows occurrence of mixing far in excess of that which can be explained by diffusion alone. This mixing, referred to as dispersion, is the result of spatial fluctuations in the velocity of the fluid as it moves through the complex network of pores within the matrix and occurs even in media where local average properties are independent of location, direction, and time.

DISPERSION THEORY

Perkins and Johnston (1963) discuss miscible flow in which the two phases have different viscosities and densities. Brigham, Reed, and Dew (1961) also discuss the effect of unequal viscosity on dispersion, while Slobod and Howlett (1964) treat the effects of unequal density displacements. We restrict ourselves here to discussions of equal viscosity, equal density, miscible displacements.

Kramers and Alberda (1953) argue from a cell mixing model that the differential equation describing the concentration for uniform, linear flow is

$$\frac{\partial\psi}{\partial t} = D\frac{\partial^2\psi}{\partial x^2} - u\frac{\partial\psi}{\partial x} \tag{1}$$

where ψ is the concentration, D, called the dispersion coefficient, is a term analogous to the molecular diffusivity, and u is an average velocity. Scheidegger (1965) arrived at this same form by applying a statistical model.

Experiments show that the degree of mixing in the direction of the velocity is different from that perpendicular to this direction, suggesting that D has tensorial properties. Nikolaevskii (1959) used the central limit theorem to arrive at a form for the dispersion equation for an instantaneous point source in convected coordinates

$$\frac{\partial\psi}{\partial t} = \sum_{\alpha=1}^{3}\frac{\partial}{\partial x_\alpha}\left(D_\alpha\frac{\partial\psi}{\partial x_\alpha}\right) \tag{2}$$

where D_α is the component of dispersion along the α axis:

$$D_\alpha = \frac{1}{2t}\overline{x_\alpha^2} \tag{3}$$

The bar in Equation (3) denotes the averaged value. Since D_α may be different for each coordinate axis, the dispersion coefficient is a second-order tensor. He shows that for isotropic media this tensor has the form

$$D_{ij} = a_{ijkl}\frac{\bar{v}_k\,\bar{v}_l}{\bar{v}} \tag{4}$$

where $\bar{v}_k$ is the component of the mean pore velocity, and a_{ijkl} is a fourth-order tensor, the dispersivity, describing the characteristic length of the medium.

For a fixed coordinate system, Equation (2) becomes

$$\frac{\partial\psi}{\partial t} = \frac{\partial}{\partial x_i}\left(D_{ij}\frac{\partial\psi}{\partial x_j}\right) - \bar{v}_i\frac{\partial\psi}{\partial x_i} \tag{5}$$

It should be noted that the velocity $\bar{v}$ appearing in Equation (5) and u in Equation (1) are not necessarily the same. $\bar{v}$ can be considered as the number average of the actual velocities occurring in the pores of the medium, while u is the average velocity given by the Dupuit-Forcheimer relationship

$$u = q/\phi \tag{6}$$

where the Darcy velocity q is an area average velocity. Guin, Kessler, and Greenkorn (1971*a*) show that these two velocities coincide only for completely uniform media. However, in most cases these two averages are very nearly equal.

Applying the results of de Josselin de Jong (1958) to uniform flow in a homogeneous, isotropic, porous medium, Bear (1961) used geometrical arguments to arrive at a tensorial form for the dispersion coefficient similar to that developed by Nikolaevskii (1959). de Josselin de Jong and Bossen (1961) show that Bear's arguments lead to the following form for the dispersion equation:

$$\frac{\partial \psi}{\partial t} = D_{ij} \frac{\partial^2 \psi}{\partial x_i \partial x_j} - \frac{\partial}{\partial x_i} (u_i \psi) \tag{6a}$$

with

$$D_{ij} = a_{ijkl} \frac{u_k u_l}{u} \tag{7}$$

Scheidegger (1961) argues that Equation (7) also applies to anisotropic media and that the eighty-one components of a_{ijkl} can then be reduced to thirty-six by applying symmetry conditions. Scheidegger (1961) then demonstrates that for isotropic media, the thirty-six remaining elements of a_{ijkl} reduce to the two obtained earlier by Nikolaevskii (1959) and Bear (1961).

Bachmat and Bear (1964) show that if a_{ijkl} is an isotropic tensor for a homogeneous, isotropic medium, then

$$a_{ijkl} = a_I \delta_{ij} \delta_{kl} + 2 a_{II} \delta_{ik} \delta_{jl} \tag{8}$$

where a_I and a_{II} are invariants depending on the pore structure of the medium. For a_{ijkl} given by Equation (8)

$$D_{ij} = a_I \delta_{ij} u + 2 a_{II} \frac{u_i u_j}{u} \tag{9}$$

Thus far it has been assumed that a_{ijkl} is independent of u, and Equation (9) indicates that the dispersion tensor will be linearly dependent on the velocity. Experiments in isotropic media by Harleman, Melhorn, and Rumer (1963), Harleman and Rumer (1963), and Lawson (1972) show, however, that the longitudinal dispersion coefficient is a function of u^n, where n is between 1.0 and 2.0.

Poreh (1965) used the theory of invariants given by Robertson (1940) to obtain a more nearly correct form for the dispersion tensor. Dimensional analysis led to the following form for the dispersion tensor in isotropic media

$$\frac{D_{ij}}{\mathfrak{D}} F_1 \delta_{ij} + F_2 \left(\frac{l^2}{\mathfrak{D}^2} \right) u_i u_j \tag{10}$$

where l is a length parameter describing the size of the pores, and $\mathfrak{D}$ is the molecular diffusivity. F_1 and F_2 are dimensionless functions of $(ul/\mathfrak{D})^2$ (and a set of dimensionless coefficients $s_1, s_2, \ldots, s_n$ which describe the geometry of the medium).

Whitaker (1967) derives the following differential equation describing diffusion and dispersion for laminar flow in porous media by volume averaging the equations of motion and continuity over a finite volume containing both fluid and solid phases:

$$\frac{\partial \bar{c}}{\partial t} + \frac{\partial}{\partial x_i} (\overline{v_i c}) = \frac{\partial}{\partial x_j} \left\{ \mathfrak{D} \left(\frac{\partial \bar{c}}{\partial x_j} + \mathfrak{R} \tau_j \right) \right\} \tag{11}$$

Here $\mathfrak{R}$ is the ratio of the surface area to volume, and τ_j is the tortuosity vector defined by

$$\tau_j = \int_{\mathfrak{A}_{(i)}} [c] n_j d\mathfrak{A} \tag{12}$$

which represents the jump in concentration at the fluid-solid interface. This led to the following form for the dispersion equation:

$$\frac{\partial \bar{c}}{\partial t} + \bar{v}_i \frac{\partial \bar{c}}{\partial x_i} = \frac{\partial}{\partial x_j} \left\{ \mathfrak{D} \left(\frac{\partial \bar{c}}{\partial x_j} + \mathfrak{R} \tau_j \right) \right\}$$

$$+ \frac{\partial}{\partial x_j} \left\{ \left(A^I_{jik} \bar{v}_i + A^{II}_{jilk} \bar{v}_i \bar{v}_l + A^{III}_{jikl} \bar{v}_i \frac{\partial \bar{c}}{\partial x_l} \right) \frac{\partial \bar{c}}{\partial x_k} \right\} \tag{13}$$

where the tensors A^I, A^{II}, A^{III} are symmetric and depend on the structure of the medium as well as the transport properties of the fluid.

Patel and Greenkorn (1970) show the Equation (13), which is derived for a general, anisotropic medium, can be put in the same form as Equation (5). However, the velocity dependence of the dispersion tensor (neglecting diffusion for an anisotropic medium) is given by

$$D_{jk} = A^I_{jik} \bar{v}_i + A^{II}_{jilk} \bar{v}_i \bar{v}_1 \tag{14}$$

where $\mathbf{A}^I$ and $\mathbf{A}^{II}$ are completely symmetric. This form does not agree with that suggested by Scheidegger (1961), Equation (7), in that a third-order as well as a fourth-order tensor is required. For isotropic media, $\mathbf{A}^I = 0$, and Equation (14) reduces to Equation (7). Whitaker's analysis indicated that in isotropic materials, the fourth-order tensor has only one distinct component as opposed to the two components suggested by Nikolaevskii (1959) and Bear (1961).

The theoretical developments discussed lead to the following form for the continuity equation for uniform flow in a homogeneous porous medium:

$$\frac{\partial \psi}{\partial t} = D_{ij} \frac{\partial^2 \psi}{\partial x_i \partial x_j} - u_i \frac{\partial \psi}{\partial x_i} \tag{15}$$

For an isotropic medium with $u_i = (u_1, 0, 0)$ and with transverse dispersion negligible, Equation (15) becomes

$$\frac{\partial \psi}{\partial t} = D_L \frac{\partial^2 \psi}{\partial x^2} - u \frac{\partial \psi}{\partial x} \tag{16}$$

The boundary conditions most frequently used in solving Equation (16) are

$$\begin{aligned} \psi(x, 0) &= 0 \quad x > 0 \\ \psi(0, t) &= 1 \quad t > 0 \\ \psi(\infty, t) &= 0 \quad t > 0 \end{aligned} \tag{17}$$

The solution for these boundary conditions according to Brenner (1962) and Ogato and Banks (1971) is

$$\psi(x, t) = \frac{1}{2} \left\{ \operatorname{erfc} \left(\frac{1 - \xi}{2\sqrt{\xi \eta}} \right) + e^{1/\eta} \operatorname{erfc} \left(\frac{1 + \xi}{2\sqrt{\xi \eta}} \right) \right\} \tag{18}$$

where

$$\xi = \frac{ut}{x}, \quad \eta = \frac{D_L}{ux}$$

It has been observed by Scheidegger (1964) that small but systematic deviations occur between experimental breakthrough curves and the predictions of Equation (18). Studies by Coats and Smith (1964) and Gershon and Nir (1969) show that the use of other boundary conditions in solving Equation (16) cannot explain these deviations. Deans (1963) proposed a three-parameter mixing cell model to explain these deviations. The three parameters in Dean's model are the length of the mixing cell, the stagnant volume fraction, and a mass transfer factor. Several other models for dispersion in porous media are discussed by Bischoff and Levenspiel (1962*a*, 1962*b*).

The single-continuum model does not consider the microscopic nature of the medium. To relate the hydrodynamic properties to the matrix structure of the medium, Scheidegger (1965) used statistical models. de Josselin de Jong, (1958) and Saffman (1959) independently developed similar statistical models based on the early work of Scheidegger (1954). Saffman's model consists of an assembly of straight, randomly oriented and uniform size pores through which a marked particle travels in a series of statistically independent steps. The probability of the particle entering a particlar pore at each step was assumed proportional to the rate at which the pore transmits fluid. Saffman (1960) discusses dispersion when the effect of molecular mixing is comparable with or greater than the macroscopic mixing.

Haring and Greenkorn (1970) extended Saffman's model to nonuniform media by assuming that the pore radii and pore lengths were randomly distributed according to the Beta distribution. From this model they derive forms for the capillary pressure, permeability, and dispersion coefficients. Pakula and Greenkorn (1971) conducted an experimental investigation based on the model of Haring and Greenkorn (1970) and found that the model was capable of describing permeabilities and dispersion coefficients. Guin, Kessler, and Greenkorn (1971*a*, *b*, 1972) develop a statistical model for porous media designed to describe both nonuniformity and anisotropy.

A third type of model, the phenomenological, is based on correlation of specific experimental observations. Harleman, Mehlhorn, and Rumer (1963) related permeability and dispersion to the particle size in unconsolidated media. They obtained the following relationships:

$$k = c d_{50}^2 \qquad (19)$$

$$D_L/\nu = \psi R_{d_{50}}^{\,n} \qquad (20)$$

$$D_L/\nu = \phi R_k^{\,n} \qquad (21)$$

where $R_{d_{50}}$ is the Reynolds number based on the 50% particle size, and R_k is the permeability Reynolds number $R_k = u\sqrt{k/\nu}$.

DISPERSION IN TRANSVERSELY ISOTROPIC SLABS

Moranville, Kessler, and Greenkorn (1977) developed theory for various types of anisotropic porous media including transversely isotropic media based on the model of Guin, Kessler, and Greenkorn (1971*a*, *b*, 1972). A medium made up of alternate layers of different porous material approximates a transversely isotropic material provided the layers are sufficiently small compared to the other dimensions of the media. In such a material, the axis of symmetry is perpendicular to the layer. For such a medium the directional dispersion coefficient is given by Moranville, Kessler, and Greenkorn (1977) as

$$\frac{D(n)}{u} = \frac{A_o + A_I \cos^2\beta + A_{II}\cos^4\beta}{[\cos^2\beta + r^2 \sin^2\beta]} \qquad (22)$$

A transversely isotropic medium is shown schematically in Figure 1. It consists of alternating layers of porous material, where the alternate layers have different permeabilities and dispersion coefficients. Consider a semi-infinite slab taken from a transversely isotropic medium in such a way that the unit normal $(\vec{n})$ to the face of the slab makes an angle β with the axis of symmetry $(\vec{\lambda})$, as shown in Figure 2. If the upper face of the slab is maintained at pressure P_0 and the lower at pressure P_L, the pressure gradient will be parallel to $\vec{n}$, and the component of the Darcy velocity parallel to $\vec{\lambda}$ will be

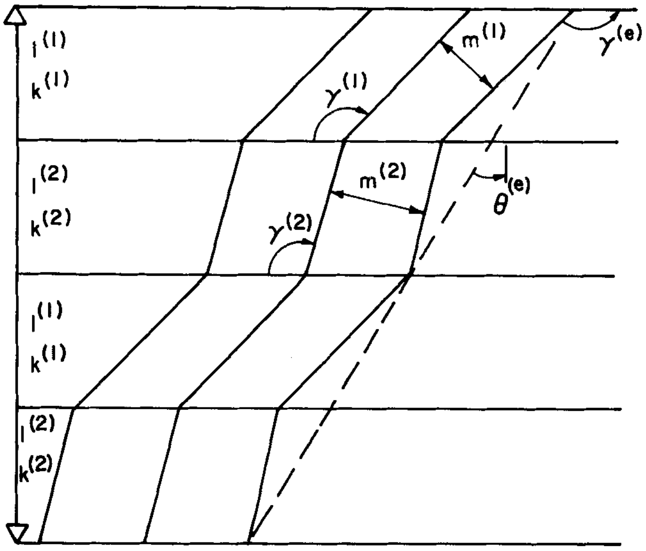

Fig. 1. Layered medium.

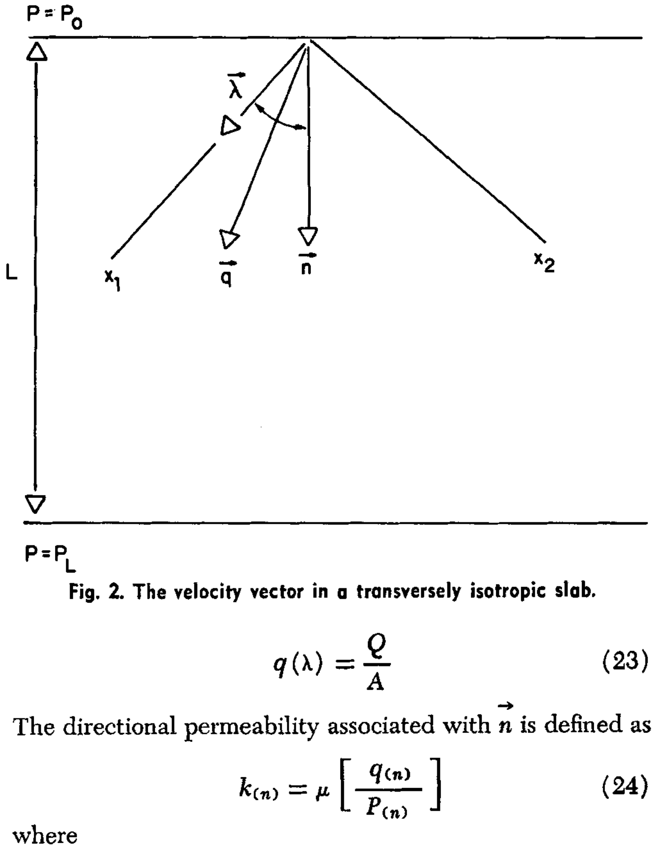

Fig. 2. The velocity vector in a transversely isotropic slab.

$$q(\lambda) = \frac{Q}{A} \qquad (23)$$

The directional permeability associated with $\vec{n}$ is defined as

$$k_{(n)} = \mu\left[\frac{q_{(n)}}{P_{(n)}}\right] \qquad (24)$$

where

$$\frac{\partial P}{\partial x_i} = n_i \frac{\Delta P}{L} = -\,n_i P_{(n)} \qquad (25)$$

In order to relate $k_{(n)}$ to the permeability tensor k_{ij}, we have

$$q_{(n)} = n_i q_i \qquad (26)$$

But

$$q_i = -\frac{1}{\mu} k_{ij} \frac{\partial P}{\partial x_j} \qquad (27)$$

so

$$q_{(n)} = (n_i k_{ij} n_j)\,\frac{P_{(n)}}{\mu} \qquad (28)$$

and

$$k_{(n)} = n_i k_{ij} n_j \qquad (29)$$

When the principal directions of k_{ij} are taken as coordinate axes, Equation (29) becomes

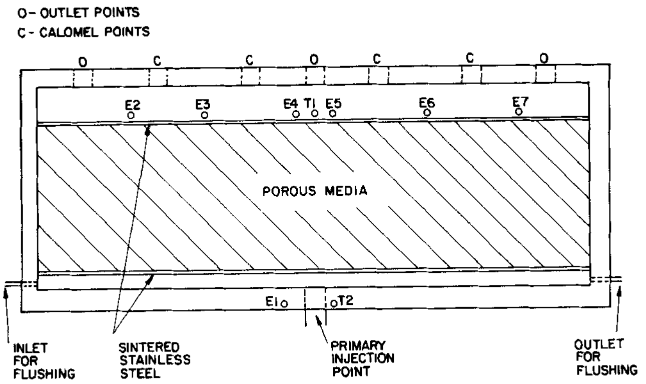

Fig. 3. Experimental model.

$$k_{(n)} = k_{11}\cos^2\beta + k_{22}\sin^2\beta \tag{30}$$

or

$$k_{(n)} = k_{22} + (k_{11} - k_{22})\cos^2\beta \tag{31}$$

By measuring $k_{(n)}$ for various values of β, the components of the permeability tensor can be determined. Experimental data for directional permeability $k_{(n)}$ obtained for the conditions of Figure 2 should plot vs. $\cos^2\beta$ as a straight line.

Consider a step change in concentration along the upper face of the slab of Figure 2. The concentration gradient in the medium will be parallel to $\vec{n}$ and may be written as

$$\frac{\partial\psi}{\partial x_i} = n_i\frac{\partial\psi}{\partial n} \tag{32}$$

$$\frac{\partial}{\partial x_i}\left(\frac{\partial\psi}{\partial x_j}\right) = n_i n_j\frac{\partial^2\psi}{\partial n^2} \tag{33}$$

$$\frac{\partial\psi}{\partial t} = D_{(n)}\frac{\partial\psi}{\partial n^2} - u_{(n)}\frac{\partial\psi}{\partial n} \tag{34}$$

Where

$$D_{(n)} = n_i D_{ij} n_j \tag{35}$$

and

$$u_{(n)} = n_i u_i = \frac{q_{(n)}}{\phi} \tag{36}$$

The boundary conditions for this system are

$$\begin{aligned} \psi &= 0 \quad \text{for} \quad t = 0, \quad 0 < x < \infty \\ \psi &= 1 \quad \text{for} \quad t > 0, \quad x = 0 \\ \psi &= 0 \quad \text{for} \quad t \geq 0, \quad x = \mathrm{L} \to \infty \end{aligned} \tag{37}$$

The solution of Equation (34) with the boundary condition (37) is

$$\psi = \frac{1}{2}\left|\operatorname{erfc}\left(\frac{1-\xi}{2\sqrt{\xi n}}\right) + \exp\left(\frac{1}{n}\right)\operatorname{erfc}\left(\frac{1+\xi}{2\sqrt{\xi n}}\right)\right| \tag{38}$$

where

$$\xi = \frac{u_{(n)}t}{L}, \quad n = \frac{D_{(n)}}{u_{(n)}L}$$

The mean seepage velocity $\vec{u}$ is

$$u_i = q_i/\phi = -\frac{1}{u\phi}k_{ij}n_j P_{(n)} \tag{39}$$

Therefore

$$u_1 = \frac{k_{11}}{\mu\phi}\frac{\Delta P}{L}\cos\beta = h_{(1)}\cos\beta \tag{40}$$

$$u_2 = \frac{k_{22}}{\mu\phi}\frac{P}{L}\sin\beta = h_{(2)}\sin\beta \tag{41}$$

$$u = |h^2_{(1)}\cos^2\beta + h^2_{(2)}\sin^2\beta|^{1/2} \tag{42}$$

For the coordinate system shown in Figure 2, the directional dispersion is

$$D_{(n)} = D_{11}\cos^2\beta + 2D_{12}\cos\beta\sin\beta + D_{22}\sin^2\beta \tag{43}$$

since

$$\frac{u_1}{u} = \frac{\cos\beta}{\sqrt{\cos^2\beta + r^2\sin^2\beta}} \tag{44}$$

$$\frac{u_2}{u} = \frac{r\sin\beta}{\sqrt{\cos^2\beta + r^2\sin^2\beta}} \tag{45}$$

$$\frac{u_{(n)}}{u} = \frac{\cos^2\beta + r\sin^2\beta}{\sqrt{\cos^2\beta + r^2\sin^2\beta}} \tag{46}$$

where

$$r = \frac{k_{22}}{k_{11}}$$

$$u_{(n)} = \frac{q_{(n)}}{\phi}$$

From Moranville, Kessler, and Greenkorn (1977)

$$\frac{D_{(n)}}{u} = a_0 + a_1\cos^2\beta + a_2\left(\frac{u_1}{u}\right) + a_3\left(\frac{u_1}{u}\right)\cos^2\beta + a_4\left(\frac{u_{(n)}}{u}\right)^2 + 2a_5\left(\frac{u_1}{u}\right)\left(\frac{u_{(n)}}{u}\right)\cos\beta \tag{47}$$

Rearranging Equation (47) and substituting Equations (44), (45), and (46), we obtain Equation (22), where

$$\begin{aligned} A_0 &= r^2(a_0 + a_4) \\ A_{\mathrm{I}} &= (a_0 + a_2) + 2r(a_4 + a_5) + r^2(a_1 - a_0 - 2a_4) \\ A_{\mathrm{II}} &= (a_1 + a_3 + a_4 + 2a_5) - 2r(a_4 + a_5) + r^2(a_4 - a_1) \end{aligned} \tag{48}$$

and the A_i's are constrained as follows:

$$a_0 + a_1 + a_2 + a_3 + a_4 + 2a_5 > 0$$

$$a_0 + a_1 > 0$$

$$a_0 + a_2 > 0$$

$$a_0 + a_4 > 0$$

$$a_4 > a_1$$

$$a_0 > 0$$

$$a_2 + a_3 + a_4 + 2a_5 > 0$$

$$a_1 + a_2 + a_3 + 2a_5 < 0$$

EXPERIMENTAL METHODS

The models used to test equation (22) consist of alternating layers of sintered glass and unconsolidated glass beads. The sintered glass used was QF-10 grade Filtros manufactured by the Electro Refractories and Abrasives Division of Ferro Corporation. It has a maximum pore diameter of 60 μm and a permeability of approximately 0.16 × 10^{-10}cm². The glass beads used were No. 203 Uni-Spheres from Microbeads Corporation. The nominal bead size distribution was 840 to 590 μ, with a permeability of 3.15 × 10^{-6} cm².

The models were contained in Lucite boxes as in Figure 3. The porous medium itself was bounded by two porous stainless steel plates. There was an entrance section (0.48 cm × 2.54 cm × 61 cm) upstream of the medium. The points labeled E1-E7 are ports through which silver-silver chloride electrodes were inserted into the model. Points T1 and T2 are ports used to measure the pressure drop.

TABLE 1. DIMENSIONS AND PACKING ANGLES FOR EXPERIMENTAL MODELS

Model No.	Angle (β)	Length (cm)	Thickness (cm)	Width (cm)
II	90 deg	14.9	2.54	60.2
III	0 deg	14.9	2.51	60.5
IV	45 deg	14.9	2.60	60.9
V	60 deg	14.9	2.60	60.9
VI	30 deg	14.9	2.59	60.9
VI	15 deg	14.9	2.54	61.0
VIII	75 deg	14.9	2.51	61.1

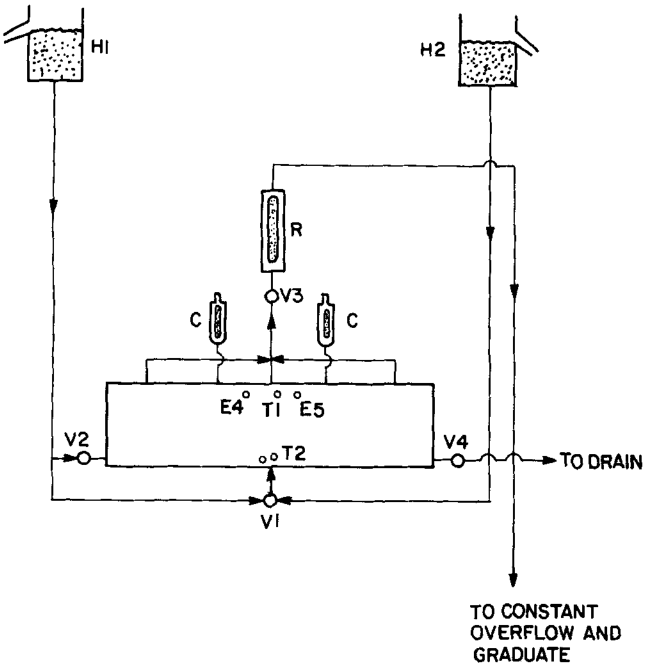

Fig. 4. Flow system.

H1	**0.20 M potassium chloride constant head feed tank**
H2	**0.20 M potassium chloride constant head feed tank**
V1	**Three-way valve**
V2-V4	**Valves**
R	**Rotameter**
E4, E5	**Silver-silver chloride electrode ports**
T1, T2	**Pressure drop taps**
C	**Calomel reference electrode**

The sintered glass was cut into 1.27 cm thick layers at the desired angle and glued into the box at 1.27 cm spacings with a silicone caulking compound. A rubber diaphragm was placed over the sintered glass pieces and the back of the box attached. The glass beads were then packed into the spaces between the sintered glass through holes drilled in the front of the box. To ensure a tight, uniform pack, the model was vibrated on a vibrating table and the diaphragm compressed against the sintered glass by air pressure of 6.89 kN/m^2. When the packing was complete, the filling holes were plugged with Lucite rods and caulking compound. The models were thoroughly flushed with carbon dioxide before they were filled with liquid to eliminate trapped gases. Table 1 gives the dimensions and packing angles for the seven models used.

Figure 4 is a schematic representation of the flow system. The fluids flowed through the medium from constant head tanks, and the flow rate was monitored by a rotameter and controlled by a valve. For dispersion measurements the flow rate was also measured volumetrically. The pressure drop across the medium was measured using a Pace-Wianko variable reluctance transducer with a 6.89 kN/m^2 diaphragm.

A potassium chloride tracer was used in the dispersion measurements. The model was filled initially with 0.02 M potassium chloride. Valve V3 was then closed, and the three-way valve V1 was switched so that 0.20 M potassium chloride was introduced. To generate a sharp step change at the inlet to the porous medium, valves V2 and V4 were opened, and 0.20 M potassium chloride allowed to flow laterally through the entrance section. Once the lower concentration fluid was thoroughly flushed from the entrance section, V2 and V4 were closed and V3 was opened, allowing the 0.20 M potassium chloride solution to flow through the model, displacing the lower concentration fluid.

TABLE 2. PERMEABILITIES OF EXPERIMENTAL MODELS

Model	β	$k \times 10^6$ (cm^2)
II	90	1.52
III	0	0.539
IV	45	0.797
V	60	1.13
VII	15	0.315
VIII	75	1.43

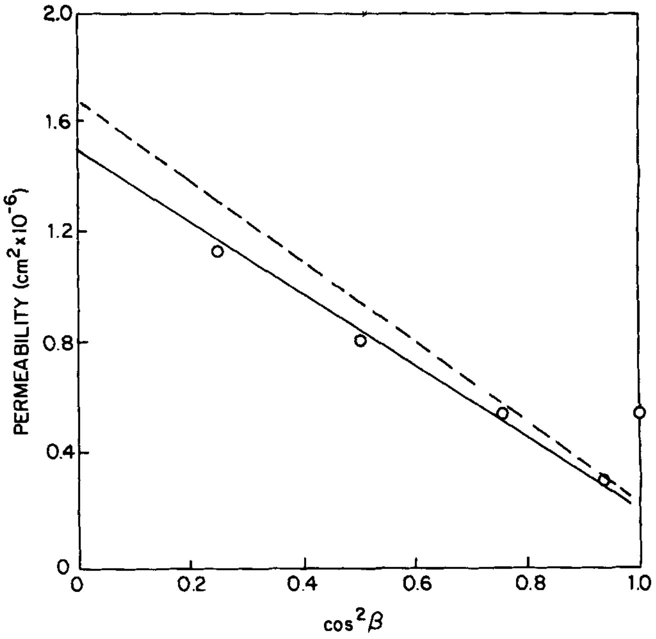

Fig. 5. Permeability data.

The concentration of the effluent from the bed was monitored at ports E4 and E5 using silver-silver chloride electrodes in conjunction with calomel reference electrodes. The two electrodes were located 1 in. to either side of the center line of the model. The potential differences between the electrodes and their corresponding reference electrodes are functions of potassium chloride concentration and were recorded from a Leeds and Northrup balancing potentiometer as functions of time. The silver-silver chloride electrodes were calibrated by measuring the developed electromotive force in potassium chloride solutions of known concentrations.

RESULTS AND DISCUSSION

Table 2 shows the measured permeabilities for the various experimental models. These permeabilities were calculated from the slope obtained on a plot of flow rate vs. pressure. The directional permeability should be a linear function of $\cos^2\beta$ according to Equation (31). Figure 5 shows $k_{(n)}$ as a function of $\cos^2\beta$. From this plot we see that with the exception of model III ($\beta = 0$ deg), the permeabilities do fall on a straight line. The broken line in Figure 5 shows the directional permeabilities predicted from permeabilities of the glass beads and sintered glass used in the experiments.

The directional dispersion coefficients were calculated by a least-squares fit of Equation (38) to the experi-

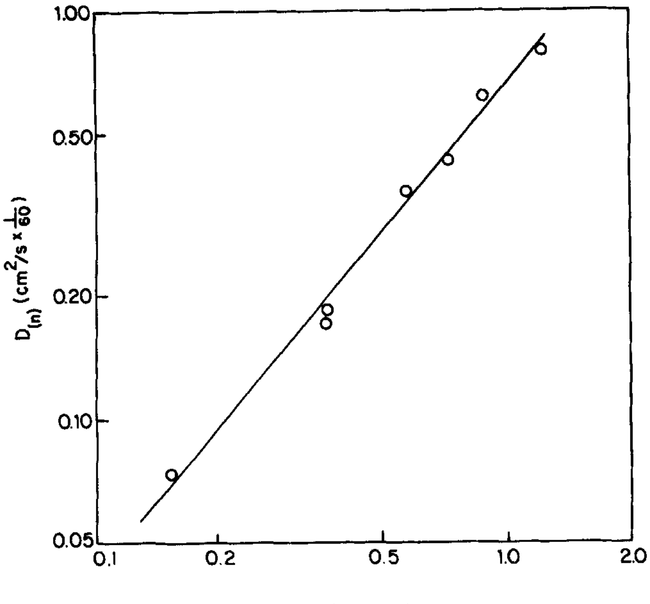

Fig. 6. $D_{(n)}$ **for model V.**

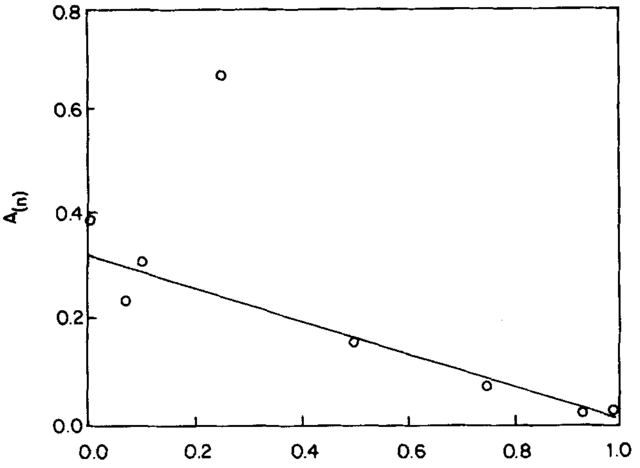

Fig. 7. $A_{(n)}$ **as a function of** β**.**

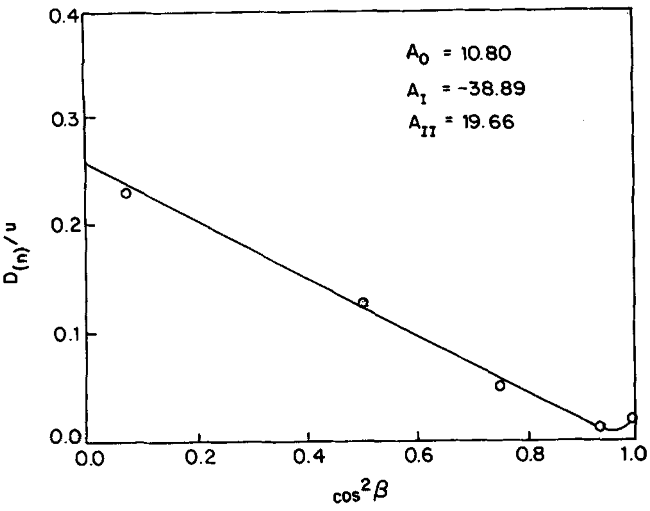

Fig. 8. Comparison between Equation (22) and experimental results.

TABLE 3. CORRELATION OF $D_{(n)}$ WITH $u_{(n)}$

Model	β	$A_{(n)}$	$N_{(n)}$
II	90 deg	0.390	0.972
III	0 deg	0.0263	1.17
IV	45 deg	0.159	0.906
V	60 deg	0.665	1.21
VI	30 deg	0.0751	0.963
VII	15 deg	0.0256	1.16
VIII	75 deg	0.236	1.13

mental breakthrough curves. The values calculated are given in the appendix. Figure 6 shows for one of the models, model V, the relationship between the directional dispersion coefficients and $u_{(n)}$, the component of the velocity along the shorter axis of the model. Each point on this figure results from determining an experimental breakthrough curve at a particular seepage velocity. If all of the curves similar to Figure 6 are plotted from the data in the appendix, the results would give about the same with the exception of model A which has a larger scatter. All the results show the relationship shown in Figure 6.

Figure 6 together with the data on the other models shows that the directional dispersion coefficient $D_{(n)}$ can be related to $u_{(n)}$ by

$$D_{(n)} = A_{(n)}[u_{(n)}]^{N_{(n)}} \tag{48a}$$

which is similar in form to the relationship between D_L and u in isotropic media. Here $A_{(n)}$ and, to a lesser extent, $N_{(n)}$ are functions of angle. The values for these parameters are listed in Table 3.

Figure 7 shows $A_{(n)}$ as a function of $\cos^2\beta$. On this plot the point corresponding to $\beta = 60$ deg (model V) appears to be highly inconsistent with the data from the other models. It seems unlikely that such a large dispersion should occur at this intermediate angle. As the flow direction approaches the direction of the layers, the possibility of channeling by the fluid parts the low permeability layers may cause an increase in the experimentally observed dispersion.

Since the boundary conditions involved in the experiments approach those of the infinite slab, Equation (22) should describe the directional dispersion coefficient. However, we note that the mathematics describes a point property (anisotropy). In order to completely verify our results, we would have to run models of different size and thinner layers to show that $D_{(n)}$ is completely independent of the physical model boundary conditions. From the permeability data, $r = 8.60$. Figure 8 shows a comparison between the data and Equation (22). The values of A_0, A_I, and A_{II} used in determining the curve in this figure were calculated from the values of $D_{(n)}/u$ for models III, IV, VII, corresponding to values of 0, 45, and 15 deg for β. Figure 8 shows that Equation (22) represents well the relation between $D_{(n)}/u$ and β.

To explain the minimum in Figure 8, it is necessary to investigate the makeup of the ordinate $D_{(n)}/u$. From Equation (48) we see that for $N_{(n)} = 1.0$

$$D_{(n)} \cong A_{(n)}u_{(n)} \tag{49}$$

$$\frac{D_{(n)}}{u} \cong A_{(n)}\frac{u_{(n)}}{u} \tag{50}$$

Figure 7 shows that $A_{(n)}$ is nearly linear in $\cos^2\beta$. However, from Equation (46), $u_{(n)}/u$ possesses a minimum with respect to $\cos^2\beta$. Therefore, it is reasonable to have a minimum in $D_{(n)}/u$ for other values of A_0, A_I, and A_{II}.

NOTATION

A = cross-sectional area
a_i = scalar invariant describing dispersion in transversely isotropic media
a_{ijkl} = dispersivity tensor
D_{ij} = dispersion tensor
$D_{(n)}$ = dispersion coefficient in n direction for slab
k = permeability
L = length
l = length
$\vec{n}$ = unit directional vector
P = pressure
Q = volumetric flow rate
$\vec{q}$ = Darcy velocity
r = ratio of major and minor axes of permeability, k_{22}/k_{11}
$\vec{u}$ = mean seepage velocity
x = position coordinate

Greek Letters

β = angle between $\vec{n}$ and $\vec{\lambda}$
γ = angle between $\vec{u}$ and the layering direction
θ = angle between $\vec{u}$ and $\vec{\lambda}$
$\vec{\lambda}$ = unit vector in direction of axis of symmetry
μ = viscosity
ν = kinematic viscosity
ϕ = porosity
ψ = dimensionless concentration

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APPENDIX: EXPERIMENTALLY DETERMINED DISPERSION COEFFICIENTS

TABLE A1. DISPERSION COEFFICIENTS FOR MODEL II

u (cm/s) × 60	D (cm^2/s) × 60
1.20	0.409
1.21	0.671
1.22	0.486
1.93	0.669
0.756	0.367
1.70	0.519
1.71	0.629
0.760	0.311
0.429	0.320
0.198	0.0768
0.473	0.372
0.203	0.0759
0.355	0.141

Table A2. Dispersion Coefficients for Models III and IV

Model III $u_{(n)}$ (cm/s) × 60	Model III $D_{(n)}$ $(\text{cm}^2/\text{s}) \times 60$	Model IV $u_{(n)}$ (cm/s) × 60	Model IV $D_{(n)}$ $(\text{cm}^2/\text{s}) \times 60$
0.739	0.0177	0.877	0.139
0.350	0.00800	0.550	0.0920
0.908	0.0224	1.20	0.199
0.154	0.00297	0.350	0.0625
1.24	0.0360	0.151	0.0291
0.901	0.0234	0.548	0.0899

Table A3. Dispersion Coefficients for Models V and VI

Model V $u_{(n)}$ (cm/s) × 60	Model V $D_{(n)}$ $(\text{cm}^2/\text{s}) \times 60$	Model VI $u_{(n)}$ (cm/s) × 60	Model VI $D_{(n)}$ $(\text{cm}^2/\text{s}) \times 60$
0.721	0.446	0.874	0.0705
0.877	0.640	0.550	0.0533
0.156	0.0732	0.357	0.0264
0.358	0.176	0.155	0.0152
1.21	0.826	1.20	0.0799
0.565	0.363	0.560	0.0551
0.362	0.185	0.739	0.0613

Table A4. Dispersion Coefficients for Models VII and VIII

Model VII $u_{(n)}$ (cm/s) × 60	Model VII $D_{(n)}$ $(\text{cm}^2/\text{s}) \times 60$	Model VIII $u_{(n)}$ (cm/s) × 60	Model VIII $D_{(n)}$ $(\text{cm}^2/\text{s}) \times 60$
0.819	0.0249	0.902	0.277
1.20	0.0309	0.566	0.106
0.548	0.0110	0.155	0.0359
0.351	0.00936	0.360	0.053
0.157	0.00304		
0.565	0.0155		

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Stochastic Models of Algal Photosynthesis in Turbulent Channel Flow

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Models have been formulated and analyzed for photosynthesis by algae in turbulent, channel flow. Analytical and computational results for different stochastic, kinematic models of algal motion have been obtained for two different rate mechanisms. The results indicate that turbulent mixing can achieve an increase in the rates and efficiencies of photosynthesis by realizing the intermittency effects. Optimum levels of turbulence are shown to exist in turbulent channel flow for the mass cultivation of algae. The methodology of this paper is relevant to photochemical reactions in general.

SCOPE

Mass cultures of algae have been of interest to engineers as a possible supplemental protein source; for the production of enzymes, vitamins, and other biochemicals; as nitrogen fixers in fields for agricultural cultivation; as photosynthetic gas exchangers in life-support systems; etc. An essential feature of algal growth is the photosynthetic activity of the cells which converts carbondioxide and water with the aid of absorbed visible radiation into cellular material and oxygen. There are two aspects of algal photosynthesis which are important to the mass cultivation of algae. First, the photosynthetic rate increases with the amount of light intensity but eventually tapers off to a maximum value. Second, photosynthesis involves light and dark reactions in sequence so that a suitable flashing light pattern may increase the efficiency of light utilization measured by the photosynthetic yield relative to the energy consumed by the culture. Since algal cultures must of necessity be dense to provide for adequate absorption of light, cells removed from the surface exposed to light would receive relatively less radiation than those at the surface. If cells in the interior must receive adequate radiation, those at the surface would have to receive more than they need. Thus a compromise must be struck between low efficiency of energy utilization at the surface and low photosynthetic rates in the interior of the culture.

In view of the advantages which may accrue from suitable intermittent light patterns, the question has been raised as to whether naturally occurring turbulence in various flow situations would serve the purpose of exposing cells in the interior to better lighting conditions and

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